Gap Equations for the Two-Band Superconductors in the Presence of Nonmagnetic Impurities in the Case $g_s, g_d, g_{sd} \neq 0$

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The effects of nonmagnetic impurities on the two-band model as proposed by Suhl, Matthias, and Walker for superconducting states of the transition metals are formally extended to the general case $g_s, g_d, g_{sd} \neq 0$. Use is made of Dyson's equation to obtain the 4×4 matrix; the order-parameter (gap-energy) equations are formally obtained for the general case. In going to the interband phonon-coupling limit $g_s = g_d = 0$, or to the intraband limit $g_{sd} = 0$, the results obtained reduce to those found by Chow. The effect of the impurity scattering on the critical temperature for the two-band superconductor in the general case is discussed.

I. INTRODUCTION

Suhl, Matthias, and Walker[1] introduced the two-band model to describe the superconducting state of the transition metals. In most of these metals, the s band and d band overlap. Also it has been known that s-d interband scattering contributes to the resistivity of the metal in the normal state.[2] They included an extra phonon-coupling term in the Hamiltonian to account for possible pair formation of electrons from the different bands in addition to the phonon-coupling terms for pairs's formation in each individual band. However, they did not include any nonmagnetic impurity scattering in their model. They determined that in the intraband limit, the interband coupling term being zero, two different transition temperatures resulted, one for each band. But in the interband limit, the intraband coupling being zero, only one transition temperature occured.

Chow[3] investigated the effects of the nonmagnetic impurity scattering on the transition temperatures. However, he considered only the limiting cases: (i) the intraband coupling limit $g_{sd} = 0$, and (ii) the interband limit $g_s = g_d = 0$. In case (i), the 4×4 matrix equations describing the system decomposed into two 2×2 matrix equations, each giving a different energy gap. In case (ii), the 4×4 matrix equations decomposed into two essentially identical 2×2 matrix equations. In the latter case, there was only one energy gap.

It is the aim of this paper to present the calculations for the general case (with all g_s, g_d, $g_{sd} \neq 0$). Use is made of the matrix form of Dyson's equation to find the 4×4 Green's function describing the superconducting state of the two-band model with nonmagnetic impurity scattering. The bare 4×4 Green's function is taken to be that describing the superconducting state without the impurity scattering. The self-energy corrections are then due to the impurity scattering. By identifying the proper elements of the 4×4 Fourier-transformed Green's function,[4] the exact expressions for the order parameter $\bar{\Delta}_s$, $\bar{\Delta}_d$, and $\bar{\Delta}_{sd}$ are obtained without going to either the interband or intraband limit. It goes without saying that $\bar{\Delta}_s$, $\bar{\Delta}_d$, and $\bar{\Delta}_{sd}$ take the same form as that given by Chow[5] in the limiting cases. The fact that the 4×4 matrix equations decompose into two (different or essentially the same, depending on whether it is the intraband or interband limit) 2×2 systems of equations does not need to be known explicity for his results to obtain.

II. TWO-BAND MODEL

The Hamiltonian for the two-band model with no impurities is written in the second quantized form

$$\begin{aligned} H_0 = \sum_\sigma \int d^3x \Big[& \psi^\dagger_{s\sigma}(x)\left(-\frac{\nabla^2}{2m_s} - \mu\right)\psi_{s\sigma}(x) \\ & + \psi^\dagger_{d\sigma}(x)\left(-\frac{\nabla^2}{2m_d} - \mu\right)\psi_{d\sigma}(x) \\ & - \tfrac{1}{2} g_s \psi^\dagger_{s\sigma}(x)\psi^\dagger_{s-\sigma}(x)\psi_{s-\sigma}(x)\psi_{s\sigma}(x) \\ & - \tfrac{1}{2} g_d \psi^\dagger_{d\sigma}(x)\psi^\dagger_{d-\sigma}(x)\psi_{d-\sigma}(x)\psi_{d\sigma}(x) \\ & - g_{sd}\psi^\dagger_{s\sigma}(x)\psi^\dagger_{d-\sigma}(x)\psi_{d-\sigma}(x)\psi_{s\sigma}(x)\Big]. \end{aligned} \tag{1}$$

The $\psi_{s\sigma}(x)$ and $\psi^\dagger_{s\sigma}(x)[\psi_{d\sigma}(x)$ and $\psi^\dagger_{d\sigma}(x)]$ are, respectively, the destruction and creation operators for the s-band (d-band) electrons with their spins specified by σ (can be either $\uparrow$ or $\downarrow$). μ is the chemical potential Fermi energy, from which the single-particle energy is measured. The phonon-induced attractions are represented by g_s, g_d, and g_{sd}. These are, respectively, the phonon-coupling constants for the s-s electron pairing, d-d electron pairing, and s-d electron pairing. They are all real numbers.

Transforming to the Heisenberg picture, we introduce the Heisenberg operators $\bar{\psi}^\dagger_{s\sigma}(x)$, $\bar{\psi}_{s\sigma}(x)$, $\bar{\psi}^\dagger_{d\sigma}(x)$, and $\bar{\psi}_{d\sigma}(x)$. These operators, of course, satisfy the commutation relation

$$-i\frac{\partial}{\partial t}\bar{\psi}(x)=[H_0,\bar{\psi}(x)]\,. \tag{2}$$

To be able to treat all the operators simultaneously, we introduce, following Chow, the four-component operators

$$\Psi(x)=\begin{pmatrix}\bar{\psi}_{s\uparrow}(x)\\ \bar{\psi}_{d\uparrow}(x)\\ \bar{\psi}^{\dagger}_{s\downarrow}(x)\\ \bar{\psi}^{\dagger}_{d\downarrow}(x)\end{pmatrix} \tag{3}$$

and $$\Psi^{0\dagger}(x)=[\bar{\psi}^{\dagger}_{s\uparrow}(x)\,\bar{\psi}^{\dagger}_{d\uparrow}(x)\,\bar{\psi}_{s\downarrow}(x)\,\bar{\psi}_{d\downarrow}(x)]\,. \tag{4}$$

Defining the 4×4 Green's function

$$\mathcal{G}^0(xx')=-i\langle T\Psi^0(x)\Psi^{0\dagger}(x')\rangle\,, \tag{5}$$

where T is the time-ordering operator and angular brackets denote the grand canonical average.

Writing out (5) explicitly, we have

$$\begin{pmatrix} G^0_{ss}(xx') & G^0_{sd}(xx') & F^0_{ss}(xx') & F^0_{sd}(xx')\\ G^0_{ds}(xx') & G^0_{dd}(xx') & F^0_{ds}(xx') & F^0_{dd}(xx')\\ -F^{0*}_{ss}(x'x) & -F^{0*}_{ds}(x'x) & -G^0_{ss}(x'x) & -G^0_{ds}(x'x)\\ -F^{0*}_{sd}(x'x) & -F^{0*}_{dd}(x'x) & -G^0_{sd}(x'x) & -G^0_{dd}(x'x)\end{pmatrix}, \tag{6}$$

where $$G^0_{ij}(xx')=-i\langle T\bar{\psi}_{i\uparrow}(x)\,\bar{\psi}^{\dagger}_{j\uparrow}(x')\rangle\,, \tag{7a}$$

$$F^0_{ij}(xx')=\langle T\bar{\psi}_{i\uparrow}(x)\,\bar{\psi}_{j\downarrow}(x')\rangle\,. \tag{7b}$$

The 4×4 Green's function satisfies the matrix equation

$$H_0(x)\,\mathcal{G}^0(xx')=\delta(x-x')\,\delta(t-t')\,, \tag{8}$$

where the operator $H_0(x)$ is the 4×4 matrix

$$\begin{pmatrix} i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m_s}+\mu & 0 & -\bar{\Delta}_s & -\bar{\Delta}_{sd}\\ 0 & i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m_d}+\mu & +\bar{\Delta}_{sd} & -\bar{\Delta}_d\\ -\bar{\Delta}^*_s & +\bar{\Delta}^*_{sd} & i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m_s}-\mu & 0\\ -\bar{\Delta}^*_{sd} & -\bar{\Delta}^*_d & 0 & i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m_d}-\mu\end{pmatrix} \tag{9}$$

and $\delta(x-x')\,\delta(t-t')$ is also an 4×4 matrix:

$$\begin{pmatrix} \delta(x-x')\delta(t-t') & 0 & 0 & 0\\ 0 & \delta(x-x')\delta(t-t') & 0 & 0\\ 0 & 0 & \delta(x-x')\delta(t-t') & 0\\ 0 & 0 & 0 & \delta(x-x')\delta(t-t')\end{pmatrix}. \tag{10}$$

The order parameters Δ_s, Δ_d, Δ_{sd} are, respectively, $g_s\langle\psi_{s\uparrow}(x)\psi_{s\downarrow}(x)\rangle$, $g_d\langle\psi_{d\uparrow}(x)\psi_{d\downarrow}(x)\rangle$, and $g_{sd}\langle\psi_{s\uparrow}(x)\psi_{d\downarrow}(x)\rangle$.

Following the convention of Baym and Kadanoff,[4] the Fourier transforms of the Green's functions are given by

$$\mathcal{G}^0(p,\omega_n)=\int_0^{-i\beta}d(t-t')\int d^3(x-x')\times\exp[-ip\cdot(x-x')-\omega_n(t-t')]\times\mathcal{G}^0(x-x',t-t'), \tag{11}$$

$$\mathcal{G}^0(x-x')=ik_BT\sum_n\int[d^3p/(2\pi)^3]\times\exp[ip\cdot(x-x')+\omega_n(t-t')]\,\mathcal{G}^0(p,\omega_n), \tag{12}$$

where $\omega_n=(2n+1)\pi/\beta$ and the integration is that corresponding to values of the exchange phonon energy being within the range $(\omega_D,-\omega_D)$, ω_D being the Debye frequency. Thus in momentum space, the matrix equations for the Green's function become

$$\begin{pmatrix} i\omega_n-\epsilon_s & 0 & -\bar{\Delta}_s & -\bar{\Delta}_{sd} \\ 0 & i\omega_n-\epsilon_d & +\bar{\Delta}_{sd} & -\bar{\Delta}_d \\ -\bar{\Delta}_s^* & +\bar{\Delta}_{sd}^* & i\omega_n+\epsilon_s & 0 \\ -\bar{\Delta}_{sd}^* & -\bar{\Delta}_d^* & 0 & i\omega_n+\epsilon_d \end{pmatrix} \begin{pmatrix} G_{ss}^0(p,\omega_n) & G_{sd}^0(p,\omega_n) & F_{ss}^0(p,\omega_n) & F_{sd}^0(p,\omega_n) \\ G_{ds}^0(p,\omega_n) & G_{dd}^0(p,\omega_n) & F_{ds}^0(p,\omega_n) & F_{dd}^0(p,\omega_n) \\ -F_{ss}^{0*}(-p,-\omega_n) & -F_{ds}^{0*}(-p,-\omega_n) & -G_{ss}^0(-p,-\omega_n) & -G_{ds}^0(-p-\omega_n) \\ -F_{sd}^{0*}(-p,-\omega_n) & -F_{dd}^{0*}(-p,-\omega_n) & -G_{sd}^0(-p,-\omega_n) & -G_{dd}^0(-p-\omega_n) \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} . \tag{13}$$

By inspection, we see that the inverse Green's function in momentum space is nothing but

$$\mathcal{G}^{0\,-1}(p,\omega_n) = \begin{pmatrix} i\omega_n-\epsilon_s & 0 & -\bar{\Delta}_s & -\bar{\Delta}_{sd} \\ 0 & i\omega_n-\epsilon_d & +\bar{\Delta}_{sd} & -\bar{\Delta}_d \\ -\bar{\Delta}_s^* & +\bar{\Delta}_{sd}^* & i\omega_n+\epsilon_s & 0 \\ -\bar{\Delta}_{sd}^* & -\bar{\Delta}_d^* & 0 & i\omega_n+\epsilon_d \end{pmatrix} . \tag{14}$$

To treat the effects of the impurity atoms being added to the metal, we introduce the interaction Hamiltonian

$$H_I = \sum_{i,\sigma} \int d^3x \{ V_s(x-R_i)\, \psi_{s\sigma}^\dagger(x)\psi_{s\sigma}(x) + V_d(x-R_i)\, \psi_{d\sigma}^\dagger(x)\psi_{d\sigma}(x) + V_{sd}(x-R_i)[\psi_{s\sigma}^\dagger(x)\psi_{d\sigma}(x) + \psi_{d\sigma}^\dagger(x)\psi_{s\sigma}(x)] \} . \tag{15}$$

The interaction terms $V_s(x-R_i)$, $V_d(x-R_i)$, and $V_{sd}(x-R_i)$ are, respectively, the potentials of any impurity atom at position R_i as felt by an s electron, a d electron, and finally the potential which causes an interband transition.

The total Hamiltonian of the system is

$$H_T = H_0 + H_I , \tag{16}$$

and so the new Heisenberg operators, e.g.,

$$\bar{\psi}_{s\sigma}'(x) = e^{iH_T t}\psi_{s\sigma}'(x)e^{-iH_T t}$$

satisfy the new commutation relation

$$-i\frac{\partial}{\partial t}\bar{\psi}_{s\sigma}'(x) = [H_T, \bar{\psi}_{s\sigma}'(x)] . \tag{17}$$

By using the same formulism detailed in Eqs. (3)–(10), we have the following matrix equation in momentum space:

$$\begin{pmatrix} i\omega_n-\epsilon_s-n_iV_s(p) & -n_iV_{sd}(p) & -\bar{\Delta}_s & -\bar{\Delta}_{sd} \\ -n_iV_{sd}(p) & i\omega_n-\epsilon_d-n_iV_d(p) & +\bar{\Delta}_{sd} & -\bar{\Delta}_d \\ -\bar{\Delta}_s^* & +\bar{\Delta}_{sd}^* & i\omega_n+\epsilon_s+n_iV_s(p) & n_iV_{sd}(p) \\ -\bar{\Delta}_{sd}^* & -\bar{\Delta}_d^* & n_iV_{sd}(p) & i\omega_n+\epsilon_d+n_iV_d(p) \end{pmatrix}$$

$$\times \begin{pmatrix} G_{ss}(p,\omega_n) & G_{sd}(p,\omega_n) & F_{ss}(p,\omega_n) & F_{sd}(p,\omega_n) \\ G_{ds}(p,\omega_n) & G_{dd}(p,\omega_n) & F_{ds}(p,\omega_n) & F_{dd}(p,\omega_n) \\ -F_{ss}^*(-p,-\omega_n) & -F_{ds}^*(-p,-\omega_n) & -G_{ss}(-p,-\omega_n) & -G_{ds}(-p,-\omega_n) \\ -F_{sd}^*(-p,-\omega_n) & -F_{dd}^*(-p,\omega_n) & -G_{sd}(-p,-\omega_n) & -G_{dd}(-p,-\omega_n) \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} . \tag{18}$$

The potentials $V_s(p)$, $V_d(p)$, and $V_{sd}(p)$ are, of course, the Fourier transform of $V_s(x-R_i)$, $V_d(x-R_i)$, and $V_{sd}(x-R_i)$, respectively. n_i is the density of the impurity atoms.

The problem is now to solve this last equation. Our approach will be to look at (14) as the inverse bare propagator in a Dyson-like matrix equation[6]:

$$\mathcal{G}^{-1}(p,\omega_n) = \mathcal{G}^{0\,-1}(p,\omega_n) - \Sigma(p,\omega_n) . \tag{19}$$

For our problem, this is a 4×4 matrix equation. The self-energy matrix $\Sigma(p,\omega_n)$ represents the correction due to the matrix interaction

$$V(p)\begin{pmatrix} -n_iV_s(p) & -n_iV_{sd}(p) & 0 & 0 \\ -n_iV_{sd}(p) & -n_iV_d(p) & 0 & 0 \\ 0 & 0 & n_iV_s(p) & n_iV_{sd}(p) \\ 0 & 0 & n_iV_{sd}(p) & n_iV_d(p) \end{pmatrix}. \tag{20}$$

III. MATRIX DYSON'S EQUATION

According to Abrikosov, Gorkov, and Dzyaloshinski,[7] when the density of impurity atoms is small, the first-order correction to the self-energy due to the impurity scattering is a constant when averaged over the position of the impurity atom. Since this constant can be included in the ground-state energy, it does not change the inverse Green's function. This fact has led Chow,[3] Markowitz and Kadanoff,[8] and others to consider the impurity scattering correction to be represented by the second Born approximation, i.e., by

$$\Sigma_{ss}(p,\omega_n) = \int d^3p'(2\pi)^{-3}\,|V_s(p-p')|^2 \times G_{ss}(p-p',\omega_n). \tag{21}$$

For our problem, the self-energy correction will, of course, be the 4×4 matrix

$$\int\frac{d^3p}{(2\pi)^3}\begin{pmatrix} -n_iV_s & -n_iV_{sd} & 0 & 0 \\ -n_iV_{sd} & -n_iV_d & 0 & 0 \\ 0 & 0 & n_iV_s & n_iV_{sd} \\ 0 & 0 & n_iV_{sd} & n_iV_d \end{pmatrix}\begin{pmatrix} G_{ss} & G_{sd} & F_{ss} & F_{sd} \\ G_{ds} & G_{dd} & F_{ds} & F_{dd} \\ -F^*_{ss} & -F^*_{ds} & -G_{ss} & -G_{ds} \\ -F^*_{sd} & -F^*_{dd} & -G_{sd} & -G_{dd} \end{pmatrix}\begin{pmatrix} -n_iV_s & -n_iV_{sd} & 0 & 0 \\ -n_iV_{sd} & -n_iV_d & 0 & 0 \\ 0 & 0 & n_iV_s & n_iV_{sd} \\ 0 & 0 & n_iV_{sd} & n_iV_d \end{pmatrix}$$

$$=\Sigma(p,\omega_n). \tag{22}$$

Carrying through the matrix multiplication, we will have the 4×4 matrix

$$\begin{pmatrix} [V_s^2G_{ss}]+[V_{sd}^2G_{dd}] & [V_{sd}^2G_{sd}] & -[V_s^2F_{ss}]-[V_{sd}^2F_{dd}] & -[V_{sd}^2F_{ds}]-[V_sV_dF_{ds}] \\ [V_{sd}^2G_{sd}] & [V_{sd}^2G_{ss}]+[V_d^2G_{dd}] & -[V_{sd}^2F_{sd}]-[V_sV_dF_{sd}] & -[V_{sd}^2G_{ss}]-[V_d^2G_{dd}] \\ [V_s^2F^*_{ss}]+[V_{sd}^2F_{dd}] & [V_{sd}^2F^*_{sd}]+[V_sV_dF^*_{sd}] & -[V_s^2G_{ss}]-[V_{sd}^2G_{dd}] & -[V_{sd}^2G_{sd}] \\ [V_{sd}^2F^*_{ds}]+[V_sV_dF^*_{sd}] & [V_{sd}^2F^*_{ss}]+[V_d^2F^*_{dd}] & -[V_{sd}^2G_{ds}] & -[V_{sd}^2G_{ss}]-[V_d^2G_{dd}] \end{pmatrix}=\Sigma\,, \tag{23}$$

where the bracket around the elements represents the integration implied in (23), e.g.,

$$[V_s^2(p')\,G_{ss}(p'-p,\omega_n)] = n_i\int d^3p'(2\pi)^{-3}V_s^2(p')\,G_{ss}(p-p',\omega_n). \tag{24}$$

Terms of the type $[V_s(p)V_d(p)G_{ss}(p,\omega_n)]$, $[V_s(p)V_d(p)F_{ss}(p,\omega_n)]$, etc., vanish when they are averaged over the position of impurity atom.[3,7]

Using the self-energy matrix $\sum$[Eq. (23)] and $\mathcal{G}^{0-1}$ [Eq. (14)] in Dyson's equation [(19)], we have for the inverse Green's function

$$\mathcal{G}^{-1}(p,\omega_n)=\mathcal{G}^{0-1}(p,\omega_n)-\Sigma$$

$$=\begin{pmatrix} i\omega_n-\epsilon_s-[V_s^2G_{ss}]-[V_{sd}^2G_{dd}] & -[V_{sd}^2G_{ds}] & -\bar{\Delta}_s+[V_s^2F_{ss}]+[V_{sd}^2F_{dd}] & -\bar{\Delta}_{sd}+[V_{sd}^2F_{ds}]+[V_dV_sF_{ds}] \\ -[V_{sd}^2G_{sd}] & i\omega_n-\epsilon_s-[V_{sd}^2G_{ss}]-[V_d^2G_{dd}] & +\bar{\Delta}_{sd}+[V_{sd}^2F_{sd}]+[V_sV_dF_{sd}] & -\bar{\Delta}_d+[V_{sd}^2F_{ss}]+[V_d^2F_{dd}] \\ -\bar{\Delta}^*_s-[V_s^2F^*_{ss}]-[V_{sd}^2F^*_{dd}] & +\bar{\Delta}^*_{sd}-[V_{sd}^2F^*_{sd}] & i\omega_n+\epsilon_s+[V_s^2G_{ss}]+[V_{sd}^2G_{dd}] & [V_{sd}^2G_{sd}] \\ -\bar{\Delta}^*_{sd}-[V_{sd}^2F^*_{ds}] & -\bar{\Delta}^*_d-[V_{sd}^2F^*_{ss}]-[V_d^2F^*_{dd}] & [V_{sd}^2G_{ds}] & i\omega_n+\epsilon_d+[V_d^2G_{dd}]+[V_{sd}^2G_{ss}] \end{pmatrix}. \tag{25}$$

Expecting that the Green's function will be of a form similar to the bare Green's function, i.e., the Green's function for the system with no impurity scattering, we make the ansatz

$$\mathcal{G}^{-1}(p,\omega_n)=\begin{pmatrix} i\tilde{\omega}_n(s)-\epsilon_s & 0 & -\tilde{\Delta}_n(s) & -\tilde{\Delta}_n(sd) \\ 0 & i\tilde{\omega}_n(d)-\epsilon_d & +\tilde{\Delta}_n(sd) & -\tilde{\Delta}_n(d) \\ -\tilde{\Delta}_n^*(s) & +\tilde{\Delta}_n^*(sd) & i\tilde{\omega}_n(s)+\epsilon_s & 0 \\ -\tilde{\Delta}_n^*(sd) & -\tilde{\Delta}_n^*(d) & 0 & i\tilde{\omega}_n(d)+\epsilon_d \end{pmatrix} . \tag{26}$$

Inverting this matrix, we have

$$G_{ss}(p\tilde{\omega}_n)=-\{[i\tilde{\omega}_n(s)+\epsilon_s][\tilde{\omega}_n^2(d)+\epsilon_d^2+\tilde{\Delta}_n^2(d)]+\tilde{\Delta}_n^2(sd)[i\tilde{\omega}_n(d)+\epsilon_d]\}/D \ , \tag{27}$$

$$G_{dd}(p,\tilde{\omega}_n)=-\{[i\tilde{\omega}_n(d)+\epsilon_d][\tilde{\omega}_n^2(s)+\epsilon_s^2+\tilde{\Delta}_n^2(s)]+\tilde{\Delta}_n^2(sd)[(i\tilde{\omega}_n(s)+\epsilon_s)]\}/D \ , \tag{28}$$

$$F_{ss}(p,\tilde{\omega}_n)=-\{\tilde{\Delta}_n^*(s)[\tilde{\omega}_n^2(d)+\epsilon_d^2+\tilde{\Delta}_n^2(d)]+\tilde{\Delta}_n^*(sd)\tilde{\Delta}_n^*(sd)\tilde{\Delta}_n(d)\}/D \ , \tag{29}$$

$$F_{dd}(p,\tilde{\omega}_n)=-\{\tilde{\Delta}_n^*(d)[\tilde{\omega}_n^2(s)+\epsilon_s^2+\tilde{\Delta}_n^2(s)]+\tilde{\Delta}_n^*(sd)\tilde{\Delta}_n^*(sd)\tilde{\Delta}_n(s)\}/D \ , \tag{30}$$

and $$F_{sd}(p,\tilde{\omega}_n)=-(\tilde{\Delta}_n^*(sd)\{[i\tilde{\omega}_n(d)-\epsilon_d][i\tilde{\omega}_n(s)+\epsilon_s]+\tilde{\Delta}_n^2(sd)\}+\tilde{\Delta}_n(sd)\tilde{\Delta}_n(d)\tilde{\Delta}_n(s))/D \ , \tag{31}$$

where the denominator D is

$$[\tilde{\omega}_n^2(s)+\epsilon_s+\tilde{\Delta}_n^2(s)][\tilde{\omega}_n^2(d)+\epsilon_d^2+\tilde{\Delta}_n^2(d)]+\tilde{\Delta}_n^2(sd)\{-\tilde{\Delta}_n^2(sd)+[i\tilde{\omega}_n(d)+\epsilon_d]$$
$$\times[i\tilde{\omega}_n(s)-\epsilon_s]+[i\tilde{\omega}_n(d)-\epsilon_d][i\tilde{\omega}_n(s)+\epsilon_s]\}+\tilde{\Delta}_n(s)\tilde{\Delta}_n(d)[\tilde{\Delta}_n(sd)]^2+\tilde{\Delta}_n^*(s)\tilde{\Delta}_n^*(d)[\Delta_n(sd)]^2 \ ,$$

and where the values $\tilde{\omega}_n(s)$, $\tilde{\omega}_n(d)$, $\tilde{\Delta}_n(s)$, $\tilde{\Delta}_n(d)$, and $\tilde{\Delta}_n(sd)$ are obtained by comparing (26) and (25). The values are

$$\tilde{\omega}_n(s)=\omega_n+n_i\int d^3k(2\pi)^{-3}V_{sd}^2G_{dd}(k,\tilde{\omega}_n)+n_i\int d^3k(2\pi)^{-3}V_{ss}^2G_{ss}(k,\tilde{\omega}_n) \ , \tag{32}$$

$$\tilde{\omega}_n(d)=\omega_n+n_i\int d^3k(2\pi)^{-3}V_{sd}^2G_{ss}(k,\tilde{\omega}_n)+n_i\int d^3k(2\pi)^{-3}V_{dd}^2G_{dd}(k,\tilde{\omega}) \ , \tag{33}$$

$$\tilde{\Delta}_n(s)=\overline{\Delta}_s+n_i\int d^3k(2\pi)^{-3}V_s^2F_{ss}(k,\tilde{\omega}_n)+n_i\int d^3k(2\pi)^{-3}V_{sd}^2F_{dd}(k,\tilde{\omega}) \ , \tag{34}$$

$$\tilde{\Delta}_n(d)=\overline{\Delta}_d+n_i\int d^3k(2\pi)^{-3}V_d^2F_{dd}(k,\tilde{\omega}_n)+n_i\int d^3k(2\pi)^{-3}V_{sd}^2F_{ss}(k,\tilde{\omega}) \ , \tag{35}$$

$$\tilde{\Delta}_n(sd)=\tilde{\Delta}_{sd}+n_i\int d^3k(2\pi)^{-3}[V_{sd}^2+V_sV_d]F_{sd}(k,\tilde{\omega}_n) \ , \tag{36}$$

where $G_{ss}(p,\tilde{\omega}_n)$, $G_{dd}(p,\tilde{\omega}_n)$, $F_{ss}(p,\tilde{\omega}_n)$, $F_{dd}(p,\tilde{\omega}_n)$, and $F_{sd}(p,\tilde{\omega}_n)$ are given by (27)–(31).

The order-parameter equations are obtained from

$$\overline{\Delta}_s(x)=-(g_s/\beta)\sum_n\int d^3p(2\pi)^{-3}F_{ss}(p,\tilde{\omega}_n) \ , \tag{37}$$

$$\overline{\Delta}_d(x)=-(g_d/\beta)\sum_n\int d^3p(2\pi)^{-3}F_{dd}(p,\tilde{\omega}_n) \ , \tag{38}$$

and $$\overline{\Delta}_{sd}(x)=-(g_{sd}/\beta)\sum_n\int d^3p(2\pi)^{-3}F_{sd}(p,\tilde{\omega}_n) \ . \tag{39}$$

Given $F_{ss}(p,\tilde{\omega}_n)$, $F_{dd}(p,\tilde{\omega}_n)$, and $F_{sd}(p,\tilde{\omega}_n)$, we have for the order-parameter equations

$$\overline{\Delta}_s(x)=(g_s/\beta)\sum_n\int d^3p(2\pi)^{-3}\{\tilde{\Delta}_n^*(s)[\tilde{\omega}_n^2(d)+\epsilon_d^2+\tilde{\Delta}_n^2(d)]+[\tilde{\Delta}_n^*(sd)]^2\tilde{\Delta}_n(d)\}/D \ , \tag{40}$$

$$\overline{\Delta}_d(x)=(g_d/\beta)\sum_n\int d^3p(2\pi)^{-3}\{\tilde{\Delta}_n^*(d)[\tilde{\omega}_n^2(s)+\epsilon_s^2+\tilde{\Delta}_n^2(s)]+[\tilde{\Delta}_n^*(sd)]^2\tilde{\Delta}_n(s)\}/D \ , \tag{41}$$

and $$\overline{\Delta}_{sd}(x)=(g_{sd}/\beta)\sum_n\int d^3p(2\pi)^{-3}(\tilde{\Delta}_n^*(sd)\{[i\tilde{\omega}_n(s)+\epsilon_s][i\tilde{\omega}_n(d)-\epsilon_d]+\tilde{\Delta}_n^2(sd)\}+\tilde{\Delta}_n(sd)\tilde{\Delta}_n(s)\tilde{\Delta}_n(d))/D \ , \tag{42}$$

where $\tilde{\omega}_n(s)$, $\tilde{\omega}_n(d)$, $\tilde{\Delta}_n(s)$, $\tilde{\Delta}_n(d)$, and $\tilde{\Delta}_n(sd)$ are given by (28)–(32). It is interesting to note that in the limit, the intraband limit $g_{sd}=0$, the order-parameter equations (40) and (41) are just these given by Chow's[3] equations [(62) and (64)]. His defining equations for the G's and $\tilde{\Delta}_n$'s [Eq. (58)–(61)] are the same as (28)–(31) in this paper if g_{sd} is set to zero. In the interband limit $g_s=g_d=0$, our result reduces to his again.

IV. BEHAVIOR OF THE CRITICAL TEMPERATURE IN THE PRESENCE OF IMPURITY SCATTERING

The effect of nonmagnetic impurity scattering on the critical temperature of the two-band superconductor can be determined exactly by solving (40)–(42) and (32)–(36). However, the evaluation of these coupled equations is difficult even when there are no impurity atoms in the superconductor. Only in the two limits, the interband and the intraband, is the evaluation of these equations

somewhat easy. To overcome the mathematical difficulties, we note that according to texts,[7] the Green's function and the pair function, which included the effects of impurity scattering, can be obtained from the corresponding equation for the pure superconductor, by making the substitution

$$\omega_n \rightarrow \eta_{\omega(n)}\omega_n, \quad \Delta \rightarrow \eta_{\omega(n)}\Delta , \tag{43}$$

where $\eta_{\omega(n)}$ is the ratio between the elements of (26) and (14).

Consequently, the basic properties of the undoped superconductor should be similar to those of the impurity-doped superconductor. This general feature has been shown by Chow in the two limiting cases. In the intraband limit, he showed that two transition temperatures still occurred when impurities were added to the superconductor. The effect of impurity scattering was to increase the lower transition temperature and possibly decrease the higher transition temperature. For the interband limit, the effect of the impurity scattering would depend on the relative strength of the interband impurity scattering to the intraband impurity scattering. The interband impurity scattering would tend to increase the critical temperature while the intraband impurity scattering would decrease the temperature.

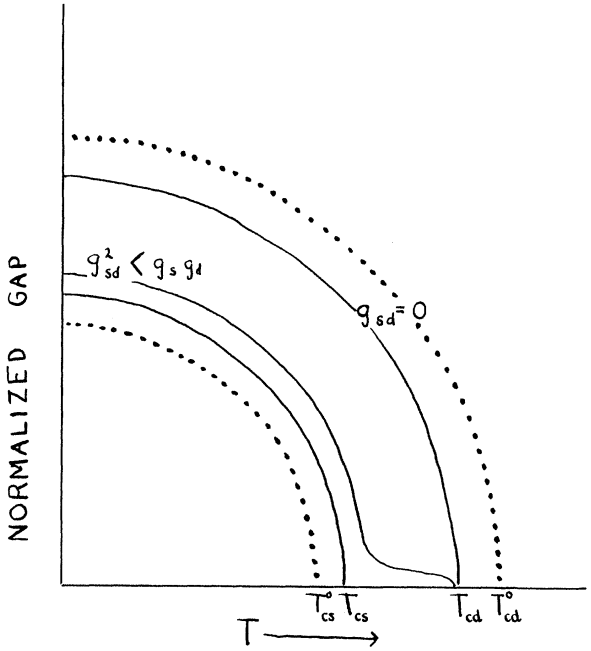

FIG. 2. Effects of nonmagnetic impurity scattering on the energy gap are shown by the solid lines; dotted lines represent the energy gaps when no impurity scattering occurs. Lower transition temperature again vanished as g_{sd}^2 becomes finite but less than $g_s g_d$.

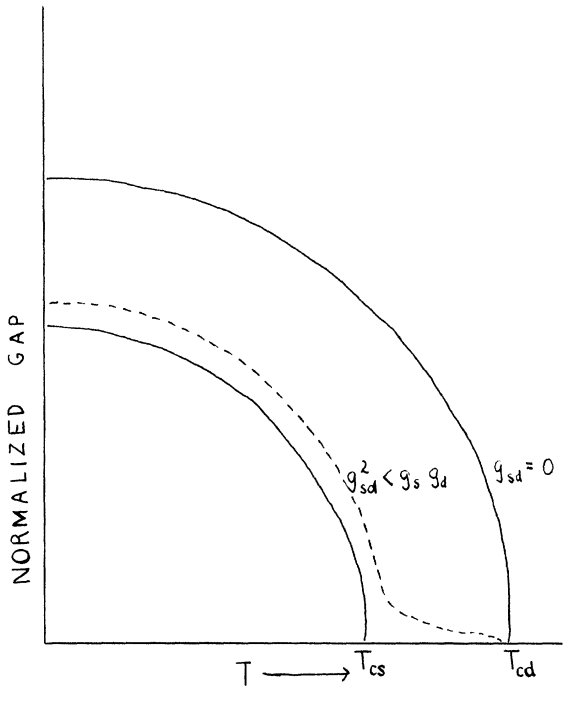

FIG. 1. When $g_{sd} = 0$ there are two transition temperatures. When g_{sd}^2 is finite but much less than $g_s g_d$, the lower transition temperature disappears in the manner shown.

To discuss qualitatively the effects of the impurity scattering on the critical temperature in the general case, we go to the results of Suhl, Matthias, and Walker.[1] The particular feature that interests us the most is shown in Fig. 1 (Fig. 2 in Ref. 1). Here we see the lower transition temperature occurring in the $g_{sd} = 0$ limit vanish as g_{sd}^2 becomes finite but being much less than $g_s g_d$. If we now include the impurity scattering effect into the discussion, we should expect the same type of behavior. Now, however, the position at which T_{cs} occurs should be shifted to a higher temperature. If Chow's conjecture on the behavior of T_{cd} is correct, the position at which the gap energy goes to zero should move to a lower temperature. The region between the two energies will therefore be narrower.

Thus, as g_{sd} becomes finite, the lower transition, even with impurities present, vanishes in the same way (see Fig. 2). The effect of the impurity scattering can be seen by looking at Eq. (32). The $\eta_{\omega(n)}$ defined in (43) takes the value

$$\eta_{\omega(n)} = 1 + n_i \int d^3p (2\pi)^{-3} (V_{sd}^2 + V_s V_d)\, \eta_{\omega(n)} F_{sd}(p, \omega_n) . \tag{44}$$

In the limit of small impurity concentration, with the value of $\eta_{\omega(n)}$ at the old critical temperature, the value at which the energy gap goes to zero in

the absence of impurity atoms is

$$\eta_{\omega}^{-1}(\zeta_n) = 1 + 2\pi n_i N_{sd}(0)\,\frac{(m_s m_d)^{1/2}}{m_s + m_d} \times \frac{1}{\omega_n}\left(\langle V_{sd}^2\rangle_{\Omega} + \langle V_s V_d\rangle_{\Omega}\right) . \quad (45)$$

Using this result in (42), we see that an additional integral occurs in the self-consistent equation at T equal the old critical temperature. The effect of this additional term is that the gap energy at the old critical temperature is not zero. Whether this change in the gap energy is positive or negative, resulting in an increased or decreased critical temperature, respectively, will depend on the same consideration as discussed by Chow in the interband phonon-coupling limit.

V. CONCLUSION

As we stated in Sec. IV, the exact behavior of the critical temperature can be obtained by solving Eqs. (40)–(42) along with (32)–(36). However, this was not attempted due to the mathematical difficulties involved in attempting to do it except in two limiting cases. It is hoped, however, that by picking the right values for the impurity scattering potentials $V_s(p)$, $V_d(p)$, and $V_{sd}(p)$ the defining set of equations [(32)–(36)] will become simple enough for the substitution (43) to be useful.

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Parameter $\kappa_2(T)$ for Strong Coupling Superconducting Alloys*

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The parameter $\kappa_2(T)$ is studied for superconducting alloys (dirty limit) in the electron-phonon model. An equation for $\kappa_2(T)$ is derived which contains the frequency-dependent part of the order parameter right at the transition from the normal to the superconducting state and which can be used as a starting point for a detailed numerical calculation. At T_c, κ_2 is expressed in terms of other measurable quantities, and at $T=0$ a rough estimate of the order of magnitude of the strong coupling corrections to κ_{2BCS} is given. It is found that the strong coupling corrections to κ_2 are very small, even for lead.

I. INTRODUCTION

In the generalization of the Ginzburg-Landau (GL) theory[1] to lower temperatures,[2,3] the well-known GL parameter κ is replaced by three different temperature-dependent parameters $\kappa_i(t)$, where $i = 1$ to 3, and $t = T/T_c$ = reduced temperature. Within the framework of the weak coupling microscopic theory, it was shown[2] that for $t \to 1$, all $\kappa_i(t) \to \kappa$. For lower temperatures, Eilenberger[4] has made a detailed calculation of $\kappa_1(t)$ and $\kappa_2(t)$ and studied in particular the influence of the mean free path on $\kappa_1(t)$ and $\kappa_2(t)$. In the extreme dirty limit, Eilenberger confirmed the result found earlier by Caroli *et al.*,[3] that $\kappa_1(0) = \kappa_2(0)$. All these calculations, however, apply to the weak coupling limit only.

For strong coupling superconductors, Eilenberger and Ambegaokar[5] (EA) presented a calculation of $H_{c2}(t)$ in the region $(1-t) \ll 1$. An extension of this theory can be combined with the well-known $H_c(t)$ calculations to derive $\kappa_1(t)$. Such a calcula-